Zeno Dynamics in Quantum Statistical Mechanics

Andreas U. Schmidt
aschmidt@math.uni-frankfurt.de
www.math.uni-frankfurt.de/~aschmidt

Fachbereich Mathematik
Johann Wolfgang Goethe Universität
Frankfurt am Main, Germany

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by

Andreas U. Schmidt

Fachbereich Mathematik
Johann Wolfgang Goethe-Universität
60054 Frankfurt am Main, Germany

and

Fraunhofer Institute Secure Telecooperation
Dolivostraße 15
64293 Darmstadt, Germany

Quantum Zeno Effect: *Evolution Impeded by Measurement*

Prepare the system in the initial state Φ_i and assume Hamiltonian evolution. Then, the short-time behavior is quadratic:

$$P_i^{(1)}(t) = |\langle \Phi_i \, , \, \exp(-\mathrm{i}tH/\hbar)\Phi_i \rangle|^2 \approx 1 - (\Delta H)^2 t^2/\hbar^2, \quad \text{where}$$

$$(\Delta H)^2 = \left\langle \Phi_i \, , H^2 \Phi_i \right\rangle - \left\langle \Phi_i \, , H \Phi_i \right\rangle^2 .$$

Perform measurements of Φ_i at times $t/N, 2t/N \ldots, t,$ modelled by $E = |\Phi_i\rangle\langle\Phi_i|$. Then

$$P_i^{(N)}(t) = \left[P_i^{(1)}(t/N)\right]^N \approx \left(1 - \tfrac{k^2 t^2}{N^2}\right)^N \overset{N \text{ large}}{\approx} \mathrm{e}^{-k^2 t^2/N} \xrightarrow{N \to \infty} 1.$$

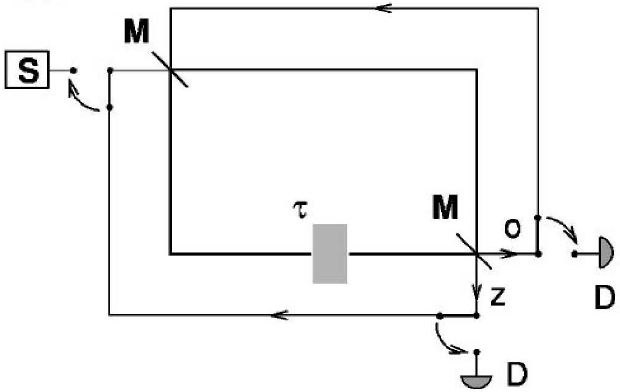
Experiments:

- (1990 Itano *et al.*) Transitions in $^9Be^+$.
- (1992 Inagaki *et al.*) Thought experiment with neutron spins.
- (2001 Wunderlich *et al.*) Transitions in $^{172}Yb^+$.

Reviews:

- H Nakazato, M Namiki, S Pascazio (1996) Internat. J. Modern Phys. B **10** 247.
- M A B Whitaker (2000) Progress in Quantum Electronics **24** 1-106.

Example: Quantum Zeno Tomography



A Mach–Zehnder interferometer can be used to detect the presence of a black sample ($\tau = 0$), without absorbing the probing particle, if it is prepared to be in the Zeno channel (Z), and detected after L rounds in the interferometer, in the limit $L \to \infty$, see P Facchi *et al.* (2002) Phys. Rev. A **66** 012110 (pictures courtesy of S. Pascazio).

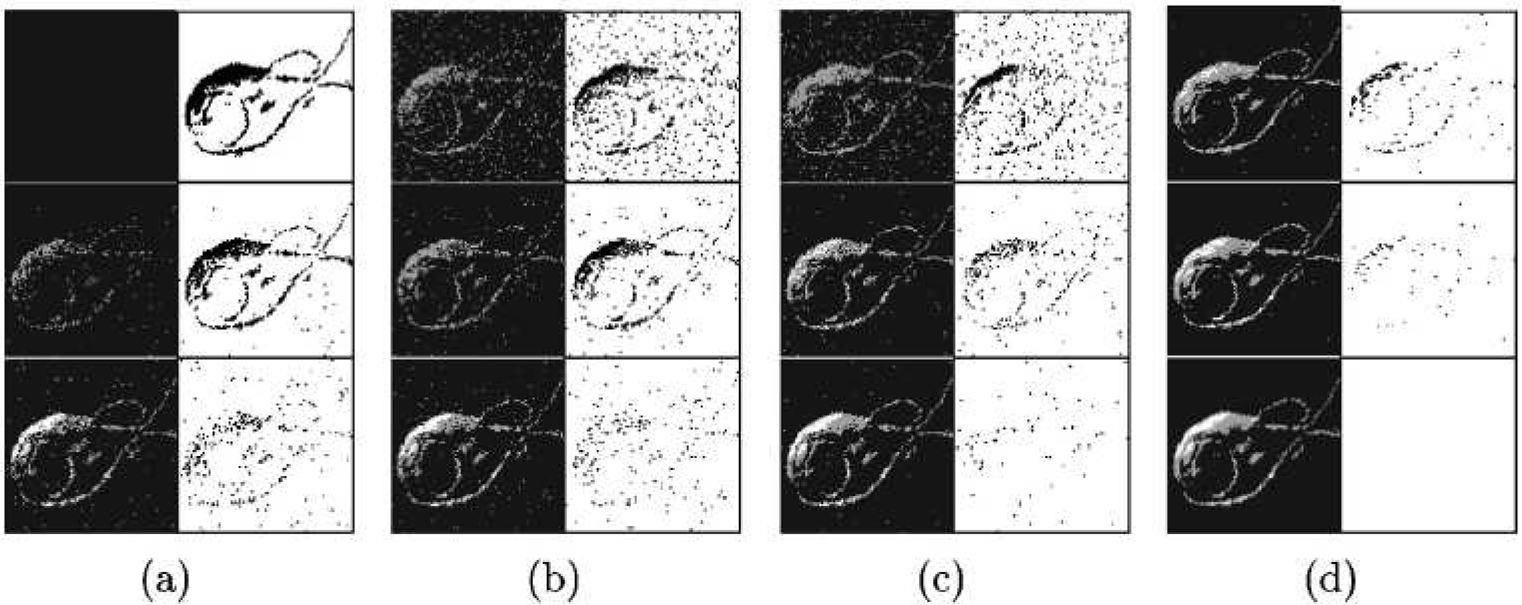


FIG. 5. Comparison of standard and Zeno tomographic techniques. In each frame, top left = reconstruction by standard technique; top right = misinterpreted pixels by the standard technique; center left = reconstruction by Zeno technique with $L=10$; center right = misinterpreted pixels by the Zeno technique with $L=10$; bottom left = reconstruction by Zeno technique with $L=165$; bottom right = misinterpreted pixels by the Zeno technique with $L=165$. The mean number of absorbed particles per pixel (irradiation) is $N_a=1.7$, 2.3, 4, and 13 for frames (a), (b), (c), and (d), respectively. The total number of particles N (total energy) scales approximately as $3N_a$, $1.8N_a$, and $6.5N_a$ for top, center, and bottom reconstructions, respectively. We used $\tau_w=0.99$, $\tau_g=0.96$, and $\tau_b=0.8$. The sample consists of 10 000 ($=100\times 100$) pixels, where white, gray, and black occur with frequencies $\alpha_w=0.02$, $\alpha_g=0.07$, and $\alpha_b=0.91$, respectively. The number of misinterpreted pixels are (top to bottom) (a) 968, 786, 315; (b) 942, 596, 212; (c) 717, 382, 68; (d) 205, 69, 0.

Zeno Dynamics in Quantum Mechanics

With $U = \mathrm{e}^{\mathrm{i}tH}$, H semibounded, E a projection on a separable Hilbert space $\mathcal{H}$:

- Does
$$W(t) \stackrel{\mathrm{def}}{=} \operatorname*{s-lim}_{n\to\infty} F_n(t), \quad \text{where } F_n(t) \stackrel{\mathrm{def}}{=} \left[EU(t/n)E\right]^n,$$
form a strongly continuous unitary group on a $\boxed{\textbf{Zeno subspace}}$ $\subset E\mathcal{H}$? That is,
- has the $\boxed{\textbf{Zeno dynamics}}$ $W(t)$ a self-adjoint generator $B = EB = BE = EBE$?

(1977 Misra & Sudarshan): Yes, if the F_n converge at all! (+ technical conditions)

The limit $n \to \infty$ is unphysical, since the duration of the measurement is always > 0, or would require an infinite amount of energy, due to time/energy uncertainty [A K Pati (1996) Phys. Lett. A **215** 7-13].

Yet, it is of conceptual interest:

- It is indicative for the appearance of the Zeno effect for finite n in models.
- Allows to identify the Zeno subspace to which the evolution will be (approximately) confined.
- The Zeno dynamics is identified as ordinary quantum dynamics with boundary conditions [Facchi, Pascazio, Scardicchio, Schulman (2001) Phys. Rev. A **65**].

Algebraic Setting of Quantum Statistical Mechanics

- The algebra of observables A (a C^*- or W^*-algebra with $\mathbb{1}$).
- A state ω on A (faithful and normal), inducing the
- GNS-representation π_ω on a Hilbert space $\mathcal{H}$, such that $\omega(\cdot) = \langle \Omega\,, \pi_\omega(\cdot)\Omega \rangle$. ($\Omega$ is cyclic and separating for $\pi_\omega(A)$.)
- The $\boxed{\textbf{von Neumann algebra}}$ $\mathcal{A} \stackrel{\text{def}}{=} \pi_\omega(A)'' = \overline{\pi_\omega(A)}^{\text{w}}$.
- The automorphic evolution $(t, A) \ni \mathbb{R} \times \mathcal{A} \longmapsto \tau_t(A) \stackrel{\text{def}}{=} U(t)AU(t)^* \in \mathcal{A}$, covariantly implemented by a strongly continuous, unitary group $U(t)$ on $\mathcal{H}$.
- The KMS condition for equilibrium states:

  ω is called $\boxed{(\tau, \beta)\textbf{-KMS state}}$, if the functions $t \longmapsto \omega(A\tau_t(B))$ extend analytically to the strip $0 < \operatorname{Im} t < \beta$, for all $A, B \in \mathcal{A}$, and

$$\omega(A\tau_{t+\mathrm{i}\beta}(B)) = \omega(\tau_t(B)A).$$

$\beta > 0$ is the $\boxed{\textbf{inverse temperature}}$. Paradigm: $\boxed{\textbf{Gibbs states}}$

$$\omega_\beta(A) \stackrel{\text{def}}{=} \frac{\mathrm{Tr}_{\mathcal{H}}\big(\mathrm{e}^{-\beta H}A\big)}{\mathrm{Tr}_{\mathcal{H}}\big(\mathrm{e}^{-\beta H}\big)}, \quad \text{for } A \in \mathcal{A},\ \dim \mathcal{H} < \infty.$$

Misra & Sudarshan's Theorem in QSM

Theorem. *Let $\mathcal{A}$, $\mathcal{H}$, U, Ω be as above, $E \in \mathcal{A}$ a projection. Assume that*

i) the following limits exist for all $A \in \mathcal{A}$:

$$W(t)A\Omega \stackrel{\text{def}}{=} \lim_{n\to\infty} F_n(t) \stackrel{\text{def}}{=} \lim_{n\to\infty} \big[EU(t/n)E\big]^n A\Omega \quad \textit{and}$$

$$W(t+\mathrm{i}\beta/2)A\Omega \stackrel{\text{def}}{=} \lim_{n\to\infty} \big[EU((t+\mathrm{i}\beta/2)/n)E\big]^n A\Omega.$$

ii) For $t \in \mathbb{R}$, $W(t)$ is weakly continuous, and $\text{w-lim}_{t\to 0} W(t) = E$.

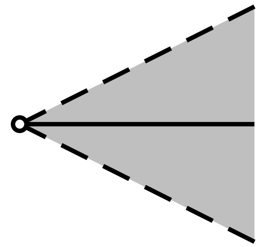
Let $\mathcal{A}_E \stackrel{\text{def}}{=} E\mathcal{A}E$ be the **reduced algebra** *and $\mathcal{H}_E \stackrel{\text{def}}{=} \overline{\mathcal{A}_E\Omega}$ the* **Zeno subspace**. *Then*

- *$W(t)$ is a strongly continuous group of unitary operators on $\mathcal{H}_E$.*
- *The action $\mathbb{R} \times \mathcal{A}_E \ni (t, A_E) \longmapsto \tau_t^E(A) \stackrel{\text{def}}{=} W(t)A_E W(t)^* \in \mathcal{A}_E$ is an automorphism group of $\mathcal{A}_E$.*
- *The vectors $W(z)A_E\Omega$, $A_E \in \mathcal{A}_E$, extend analytically to the strip $0 < \operatorname{Im} z < \beta/2$ and are continuous on its boundary.*

The proof follows [B Misra, E C G Sudarshan (1977) J. Math. Phys. **18**] using complex analysis. Differences: Holomorpy in a strip not in the upper halfplane, and $U(z)$ are unbounded (but with nice common core $\mathcal{A}\Omega$).

Comparison of Results by Analyticity Domains

- **Sectorial Semigroups (Kato 1978):** For semigroups e^{-zA}, holomorphic in a sector $\Sigma = \left\{\mathbb{C} \ni z \neq 0 \,\middle|\, |\arg z| < \theta, 0 < \theta \leq \pi/2\right\}$, the Zeno limit always exists in the interior of Σ, for every orthogonal projection E on the Hilbert space.



- **Quantum Mechanics (Misra/Sudarshan):** In this case, one needs the Zeno limit on the boundary $\mathbb{R}$ of the upper halfplane, where $U(z) = \mathrm{e}^{\mathrm{i}zH}$ is holomorphic. The above result only guarantees it for $\mathrm{Im}\, z > 0$.

- **Quantum Statistical Mechanics (AUS):** The Zeno limit is needed on the boundary of a strip in $\mathbb{C}$. No sectorial result can be used.

An Effective Condition for Zeno Dynamics

Insertion of $\mathbb{1} = E^\perp + E$ into $F_1(t) = EU(t)E$ yields

$$F_1(t) = EU(t/2)(E^\perp + E)U(t/2)E = F_2(t) + (EU(t/2)E^\perp)(E^\perp U(t/2)E).$$

The **asymptotic Zeno condition** : $\quad \left\| E^\perp U(t) E \right\| = O(t), \quad \text{as } t \longrightarrow 0. \qquad \text{(AZC)}$

(precisely: $\left\| E^\perp U(\zeta) E A\Omega \right\| = C\|A\|\zeta$ for ζ, $\operatorname{Im}\zeta \geq 0$, and $|\zeta| < r_0$ for a fixed $r_0 > 0$.)
Iterating the above calculation, one estimates $\left\| \big(F_m(t) - F_{nm}(t)\big)\Phi \right\|$, $\Phi \in \mathcal{H}$, and using the triangle inequality it follows from (AZC), for $m > n$ large enough,

$$\left\| \big(F_n(t) - F_m(t)\big)\Phi \right\| \leq \frac{C\|\Phi\|t^2}{n}.$$

- $\Longrightarrow$ The $F_n(t)$, $t \in \mathbb{R}$, form a strong Cauchy sequence.
- $\Longrightarrow$ $F_n(t)$ converges strongly to $W(t)$.
- $\Longrightarrow$ The convergence is uniform for $t \in K$, $K \Subset \mathbb{R}$ compact.
- $\Longrightarrow$ $W(t)$ is strongly continuous for t in compacts.
- $\Longrightarrow$ $W(0) = E$ ($\Longleftarrow F_n(0) = E,\ \forall n$).
- $\Longrightarrow$ $W(\zeta)A\Omega$, $A \in \mathcal{A}$, exists for $0 \leq \operatorname{Im}\zeta \leq \beta/2$, is holomorphic in the interior and continuous on the boundary.

An Existence Theorem for Zeno Dynamics

Theorem. *Let* $(\mathcal{A}, \tau, \omega, \beta > 0, \pi_\omega, \mathcal{H}, \Omega, U, E)$ *as above.*
Assume that (U, E) *satisfies (AZC) for* $\mathcal{A}$*:*

$$\left\| E^\perp U(\zeta) E A \Omega \right\| = C \cdot \|A\| \cdot \zeta$$

is valid for ζ *with* $|\zeta| < r_0$ *for some fixed* $r_0 > 0$ *and* $\operatorname{Im} \zeta \geq 0$*. Then*

i) the strong operator limits $W(t) \stackrel{\text{def}}{=} \text{s-lim}_{n\to\infty} \left[E U(t/n) E \right]^n$ *exist,*

ii) form a strongly continuous group of unitary operators on the Zeno subspace $\mathcal{H}_E \stackrel{\text{def}}{=} \overline{\mathcal{A}_E \Omega} \stackrel{\text{def}}{=} \overline{E \mathcal{A} E \Omega} \subset E\mathcal{H}$*,*

iii) and the group $W(t)$ *induces an automorphism group* τ^E *of* $\mathcal{A}_E$*.*

iv) The vectors $W(z) A_E \Omega$*,* $A_E \in \mathcal{A}_E$*, extend analytically to the strip* $0 < \operatorname{Im} z < \beta/2$ *and are continuous on its boundary.*

- The (AZC) renders applicable the methods of perturbation theory!

Example: Non-Return to Equilibrium

Return to equilibrium: [D W Robinson (1973) Comm. Math. Phys. **31** 171, Theorem 2]

Let $\mathcal{A}$ be $\boxed{\textbf{asymptotically abelian}}$, *i.e.*, $\|[A, \tau_t(B)]\| \xrightarrow{|t|\to\infty} 0$, $A, B \in \mathcal{A}$.
Let $P \in \mathcal{A}$ be a bounded perturbation, analytic for τ. Then,
for every (τ^P, β)-KMS state ω^P, the limits $\omega_\pm \stackrel{\mathrm{def}}{=} \lim\limits_{t\to\pm\infty} \omega^P \circ \tau_t$, are (τ, β)-KMS states.

Assume common, covariant implementations U, U^P of τ, τ^P on $\mathcal{H}$ (*loc. cit.* Thm. 1).

$$\Longrightarrow \quad U(t) = U^P(t) + \sum_{n\geq 1} \int_0^t \mathrm{d}t_1 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n U^P(t_1) P U^P(t_2 - t_1) P \cdots P U^P(t - t_n),$$

where the n-th term in the sum is bounded by $\|P\|^n t^n / n!$.

Let the system be prepared in a τ^P-invariant state $\Phi^P \in \mathcal{H}$, and $E = |\Phi^P\rangle\langle\Phi^P|$.

$$\Longrightarrow \quad \|E^\perp U(t) E\| \leq \underbrace{\|E^\perp U^P(t) E\|}_{=0} + t \cdot \underbrace{\|E^\perp U^{(1)}(t) E\|}_{\leq \|P\|} + O(t^2 \|P\|^2 / 2) = O(t).$$

$\Longrightarrow$ The Zeno dynamics exists. The system, evolving under τ^E, remains in the state Φ^P. **Return to equilibrium is prevented** .

$\Longrightarrow$ This effect generalizes to projections onto τ^P-invariant subspaces ($\tau^P(E) = E$).

Example: Domains of Quantum Spin Systems 1/2

Context: A $\boxed{\textbf{lattice}}$ $\mathbb{X} \stackrel{\text{def}}{=} \mathbb{Z}^d$, $\boxed{\textbf{local Hilbert spaces}}$ $\mathcal{H}_X \stackrel{\text{def}}{=} \bigotimes_{x \in X} \mathcal{H}_x, \dim \mathcal{H}_x = D < \infty$;

A $\boxed{\textbf{interaction}}$ $\Phi \colon \mathbb{X} \supset X \mapsto \mathcal{A}(X) \subset \mathcal{B}(\mathcal{H}_X)$ induces $\boxed{\textbf{local Hamiltonian dynamics}}$

$$H_\Phi(\Lambda) \stackrel{\text{def}}{=} \sum_{X \subset \Lambda} \Phi(X), \quad U_\Lambda(t) \stackrel{\text{def}}{=} \mathrm{e}^{\mathrm{i}tH_\Phi(\Lambda)}, \quad \text{for } \Lambda \Subset \mathbb{X} \text{ bounded.}$$

For $\Lambda \subset \Lambda' \Subset \mathbb{X}$ the $\boxed{\textbf{local surface interaction}}$ is

$$W_\Phi(\Lambda; \Lambda') \stackrel{\text{def}}{=} \sum \big\{ \Phi(X) \,\big|\, X \subset \Lambda',\ X \cap \Lambda \neq \varnothing,\ X \cap \Lambda' \setminus \Lambda \neq \varnothing \big\}.$$

It induces a decomposition of the Hamilonian

$H_\Phi(\Lambda') = H_\Phi(\Lambda' \setminus \Lambda) + H_\Phi(\Lambda) + W_\Phi(\Lambda; \Lambda'), \quad$ such that $\big[H_\Phi(\Lambda' \setminus \Lambda), H_\Phi(\Lambda)\big] = 0.$

From perturbation theory follows the uniform estimate

$$U_{\Lambda'}(t) = U_{\Lambda' \setminus \Lambda}(t) U_\Lambda(t) + \int_0^t U_{\Lambda' \setminus \Lambda}(\tau) U_\Lambda(\tau) W_\Phi(\Lambda; \Lambda') \mathrm{d}\tau + O(t^2). \qquad (\dagger)$$

Example: Domains of Quantum Spin Systems 2/2

Choose an arbitrary vector state $\Phi_\Lambda \in \mathcal{H}_\Lambda$. Let φ_Λ be the associated pure state.
Define the projector $E_{\varphi_\Lambda;\Lambda'} \stackrel{\text{def}}{=} \mathbb{1}_{\Lambda'\setminus\Lambda} \otimes P_{\Phi_\Lambda}$, on $\mathcal{H}_{\Lambda'} = \mathcal{H}_{\Lambda'\setminus\Lambda} \otimes \mathcal{H}_\Lambda$,
where P_{Φ_Λ} is the projector onto the one-dimensional subspace generated by Φ_Λ in $\mathcal{H}_\Lambda$

$\Longrightarrow P_\Phi(\Lambda;\Lambda') \stackrel{\text{def}}{=} H_\Phi(\Lambda) + W_\Phi(\Lambda;\Lambda')$ is a bounded local perturbation,

$\Longrightarrow \varphi_\Lambda$ is invariant under the dynamics generated by $H_\Phi(\Lambda') - P_\Phi(\Lambda;\Lambda')$

$\Longrightarrow$ The local Zeno dynamics exists: $W_{\varphi_\Lambda;\Lambda'}(t) = \lim_{n\to\infty}\left[E_{\varphi_\Lambda;\Lambda'} U_{\Lambda'}(t/n) E_{\varphi_\Lambda;\Lambda'}\right]^n$.

The thermodynamic limit $\Lambda' \to \infty$: If the $\boxed{\textbf{global surface interaction}}$

$$W_\Phi(\Lambda) \stackrel{\text{def}}{=} \sum\left\{\Phi(X) \,\middle|\, X\cap\Lambda \neq \varnothing,\ X\cap\Lambda^c \neq \varnothing\right\} = \lim_{\Lambda'\to\infty} W_\Phi(\Lambda;\Lambda')$$

is bounded, then the local Zeno limits $n \to \infty$ are uniform in Λ', and therefore

$$W_{\varphi_\Lambda}(t) = \lim_{n\to\infty}\left[E_{\varphi_\Lambda} U(t/n) E_{\varphi_\Lambda}\right]^n = \lim_{\Lambda'\to\infty} W_{\varphi_\Lambda;\Lambda'}(t),$$

where $E_{\varphi_\Lambda} \stackrel{\text{def}}{=} \lim_{\Lambda'\to\infty} E_{\varphi_\Lambda} = \mathbb{1}_{\Lambda^c} \otimes P_{\Phi_\Lambda}$,

and $U(t) \stackrel{\text{def}}{=} \lim_{\Lambda'\to\infty} U_{\Lambda'}(t)$ is the global dynamics.

Explicit Form of the Zeno Generator

Proposition. *(AZC)* $\Longrightarrow U_E(t) \stackrel{\text{def}}{=} \mathrm{e}^{\mathrm{i}tEHE} = \lim_{n\to\infty} \left[E\mathrm{e}^{\mathrm{i}t/nH}E\right]^n \stackrel{\text{def}}{=} W(t)$ *on* $\mathcal{H}_E$.

Sketch of proof: Let $\Psi_E \in \mathcal{A}_E\Omega$ be analytic for H, and assume from now on that those vectors form a dense set in $\mathcal{H}_E$ (excludes pathologies).
By a power series expansion one gets for σ, τ small enough

$$\left\|\big(U_E(\tau) - EU(\tau)E\big)U_E(\sigma)\Psi_E\right\| \leq \tau^2 \cdot C_{\Psi_E,\sigma} < \infty.$$

$$\|F_n(t)\Psi_E - U_E(t)\Psi_E\| = \left\|\left[EU(t/n)E\right]^n\Psi_E - \left[EU_E(t/n)E\right]^n\Psi_E\right\|$$

$$\leq \sum_{i=1}^n \left\|\left\{\left[EU(t/n)E\right]^{n-i}\big(EU(t/n)E - EU_E(t/n)E\big)\left[EU_E(t/n)E\right]^{i-1}\right\}\Psi_E\right\|$$

$$\leq \sum_{i=1}^n \left\|\big(U_E(t/n) - EU(t/n)E\big)U_E(t(i-1)/n)\Psi_E\right\|$$

$$\leq \sum_{i=1}^n \left(\frac{t}{n}\right)^2 \cdot \sup_{|\sigma|\leq|t|} C_{\Psi_E,\sigma} = \frac{t^2 C'_{\Psi_E,t}}{n} \qquad \Longrightarrow W(t)\Psi_E = U_E(t)\Psi_E. \qquad \square$$

A Result by Exner and Ichinose

In the QM case ($\mathcal{H}$ separable, H self-adjoint & semibounded, E orthogonal projection), a recent result (P Exner, T Ichinose math-ph/0302060) uses properties of the Zeno generator as a criterion for the existence of Zeno dynamics.

Let $H_E \stackrel{\text{def}}{=} (H^{1/2}E)^*(H^{1/2}E)$ be the self-adjoint operator associated with the quadratic form $H^{1/2}E$ on the form domain $D(H^{1/2}E)$. H_E is a self-adjoint extension of the (generally non-closed) operator EHE. H_E is defined, and acts nontrivially, on a closed subspace of $E\mathcal{H}$.

Theorem. *If* $\boxed{H_E \textbf{\textit{ is densely defined on the whole space }} \mathcal{H}}$ *then*

$$\operatorname*{s-lim}_{n\to\infty}\left[E\mathrm{e}^{-itH/n}E\right]^n = \operatorname*{s-lim}_{n\to\infty}\left[E\mathrm{e}^{-itH/n}\right]^n = \operatorname*{s-lim}_{n\to\infty}\left[\mathrm{e}^{-itH/n}E\right]^n = \mathrm{e}^{-itH_E}E$$

uniformly on every compact interval in $\mathbb{R}$.

Although only applicable to semibounded Hamiltonians, this result yields a much sharper condition for the existence of Zeno dynamics than AZC, and a more general characterization of the Zeno dynamics: We had to pose (besides AZC) the stronger condition that $\mathcal{A}_E\Omega$ contains a dense set of analytic elements for *the original Hamiltonian H*.

Gibbs Equilibria for Zeno Dynamics

Corollary. *If (U, E) satisfies (AZC) for $\mathcal{A}$, then, for every $\beta > 0$, the set of (τ^E, β)-KMS states of $\mathcal{A}_E$ coincides with the set of $(\widehat{\tau}^E, \beta)$-KMS states, where $\widehat{\tau}^E$ is induced by U_E.*

Local Gibbs states in quantum spin systems: For $A \in \mathcal{A}(\Lambda)$, $A_{E_\Lambda} \in \mathcal{A}(\Lambda)_{E_\Lambda}$,

$$\omega_\Lambda(A) \stackrel{\text{def}}{=} \frac{\mathrm{Tr}_{\mathcal{H}_\Lambda}\left(\mathrm{e}^{-\beta H(\Lambda)} A\right)}{\mathrm{Tr}_{\mathcal{H}_\Lambda}\left(\mathrm{e}^{-\beta H(\Lambda)}\right)} \quad \rightsquigarrow \quad \omega_{E_\Lambda}(A_{E_\Lambda}) \stackrel{\text{def}}{=} \frac{\mathrm{Tr}_{\mathcal{H}_{E_\Lambda}}\left(\mathrm{e}^{-\beta E_\Lambda H(\Lambda) E_\Lambda} A_{E_\Lambda}\right)}{\mathrm{Tr}_{\mathcal{H}_{E_\Lambda}}\left(\mathrm{e}^{-\beta E_\Lambda H(\Lambda) E_\Lambda}\right)}$$

- A **thermodynamic limit point** of a net of local states ω_Λ over $\mathcal{A}(\Lambda)$, is the weak* limit of the net of extensions ω_Λ^G of ω_Λ to $\mathcal{A}$.
- If the local dynamics τ^Λ generated by $H(\Lambda)$ converges uniformly to an automorphism group of $\mathcal{A}$, then every thermodynamic limit point is a (τ, β)-KMS state.

Corollary. *Let $\beta > 0$. Let $\Lambda_\alpha \to \infty$ be such that the local dynamics converges uniformly, and the net of local Gibbs states ω_{Λ_α} has a thermodynamic limit point. If E_{Λ_α} converges in norm to a projection E in $\mathcal{A}$ satisfying (AZC), then $\omega_E(A_E) \stackrel{\text{def}}{=} \lim_\alpha \omega^G_{E_{\Lambda_\alpha}}(A_E)$ is a (τ^E, β)-KMS state on $\mathcal{A}_E$.*

Domains of Quantum Spin Systems Revisited

Let the interaction be such that the conditions of the Corollary are satisfied. The local projections were $E_{\varphi_\Lambda;\Lambda'} \stackrel{\text{def}}{=} \mathbb{1}_{\Lambda'\setminus\Lambda} \otimes P_{\Phi_\Lambda}$, $\Phi_\Lambda \in \mathcal{H}_\Lambda$.

We obtain a $(\tau^{E_{\varphi_\Lambda}}, \beta)$-KMS state on $\mathcal{A}_{E_{\varphi_\Lambda}}$ by

$$\omega_{E_{\varphi_\Lambda}}(A_{E_{\varphi_\Lambda}}) \stackrel{\text{def}}{=} \lim_{\Lambda' \to \infty} \frac{\mathrm{Tr}_{\mathcal{H}_{\Lambda'}}\big(\exp(-\beta\, \mathbb{1}_{\Lambda'\setminus\Lambda} \otimes P_{\Phi_\Lambda}\, H(\Lambda')\, \mathbb{1}_{\Lambda'\setminus\Lambda} \otimes P_{\Phi_\Lambda}) A_{E_{\varphi_\Lambda;\Lambda'}}\big)}{\mathrm{Tr}_{\mathcal{H}_{\Lambda'}}\big(\exp(-\beta\, \mathbb{1}_{\Lambda'\setminus\Lambda} \otimes P_{\Phi_\Lambda}\, H(\Lambda')\, \mathbb{1}_{\Lambda'\setminus\Lambda} \otimes P_{\Phi_\Lambda})\big)}$$

where $A_{E_{\varphi_\Lambda;\Lambda'}} \in \mathcal{A}_{E_{\varphi_\Lambda;\Lambda'}}$ converges in $\mathcal{A}$ to $A_{E_{\varphi_\Lambda}}$.

Assume decompositions $\mathcal{A} = \mathcal{A}_{\Lambda^c} \otimes \mathcal{A}_\Lambda$, and $H = \sum_i H_{\Lambda^c,i} \otimes H_{\Lambda,i}$.

Then $\omega_{E_{\varphi_\Lambda}}$ is the Gibbs equilibrium state for the **averaged Hamiltonian**

$$\mathbb{E}^{\varphi_\Lambda}(H) = \sum_i H_{\Lambda^c,i} \cdot \varphi_\Lambda(H_{\Lambda,i})$$

This state is also the strong coupling ($\lambda \to \infty$) limit of the Gibbs equilibria for

$$H_\lambda = H + \lambda \mathbb{1}_{\Lambda^c} \otimes P_{\Phi_\Lambda}$$

[M Fannes, R F Werner (1995) Helv. Phys. Acta **68** 635]

Example: The X-Y Model 1/3

A spin chain over $\mathbb{X} = \mathbb{Z}$, with $\mathcal{H}_x = \mathbb{C}^2$, with observable algebras $\mathcal{A}_{[n,m]}$, $n \leq m \in \mathbb{Z}$ generated by $\boxed{\textbf{creation}}$ and $\boxed{\textbf{annihilation}}$ operators a_x, a_x^*, $n \leq x \leq m$, satisfying CAR

$$[a_x, a_y] = 0 = [a_x, a_y^*],\ x \neq y, \quad \{a_x, a_x^*\} = 1,\ \{a_x, a_x\} = 0.$$

Global observable algebra $\mathcal{A} \stackrel{\text{def}}{=} \overline{\bigcup_{n \in \mathbb{N}} \mathcal{A}_{[-n,n]}}$.

Local Hamiltonians $H_{[n,m]} \stackrel{\text{def}}{=} \dfrac{J}{2} \sum_{x=n}^{m-1} \left(a_x^* a_{x+1} + a_{x+1}^* a_x\right) + h \sum_{x=n}^{m} a_x^* a_x$.

Global dynamics $\tau_t(\cdot) \stackrel{\text{def}}{=} \lim_{n \to \infty} \mathrm{e}^{\mathrm{i}tH_{[-n,n]}} \cdot \mathrm{e}^{-\mathrm{i}tH_{[-n,n]}}$.

$(\mathcal{A}, \tau)$ is a C^*-dynamical system with unique (τ, β)-KMS state given by the weak-* limit of any increasing net of local Gibbs states, and $(\mathcal{A}, \tau)$ is asymptotically abelian.

Choose a state $\rho_0 \in \mathcal{H}_0$ and set $E_{\rho_0} \stackrel{\text{def}}{=} \mathbb{1}_{\mathcal{H}_{[-\infty,-1]}} \otimes P_{\rho_0} \otimes \mathbb{1}_{\mathcal{H}_{[1,\infty]}}$.

The interaction range is one, and E_{ρ_0} acts local $\Longrightarrow$ the Zeno dynamics $\tau^{E_{\rho_0}}$ exists.

Example: The X-Y Model 2/3

$\Longrightarrow$ The Zeno Hamiltonian decomposes $E_{\rho_0} H E_{\rho_0} = \underbrace{H_-^{\rho_0}}_{\text{acts on: } \mathcal{H}_{[-\infty,-1]}} + \underbrace{H_0^{\rho_0}}_{\mathcal{H}_0} + \underbrace{H_+^{\rho_0}}_{\mathcal{H}_{[1,\infty]}}$.

Explicitly, with $H_{[1,\infty]} = \lim_{n\to\infty} H_{[1,n]}$:

$$H_0^{\rho_0} = h\rho_0(a_0^* a_0),\ H_+^{\rho_0} = \frac{J}{2}\big(\overline{\rho_0(a_0)}a_1 + \rho_0(a_0)a_1^*\big) + H_{[1,\infty]},\ H_-^{\rho_0} \text{ similar.}$$

Lateral Gibbs states: $\omega^{\pm}_{\rho_0,\beta}(A_\pm) \stackrel{\text{def}}{=} \dfrac{\mathrm{Tr}_{\mathcal{H}_{[1,\infty]}}\big(\mathrm{e}^{-\beta H_\pm^{\rho_0}} A_\pm\big)}{\mathrm{Tr}_{\mathcal{H}_{[1,\infty]}}\big(\mathrm{e}^{-\beta H_\pm^{\rho_0}}\big)},\ A_\pm \in \mathcal{A}_\pm \stackrel{\text{def}}{=} \begin{cases}\mathcal{A}_{[1,\infty]} \\ \mathcal{A}_{[-\infty,-1]}\end{cases}$

All Zeno observables $A_{E_{\rho_0}} \in \mathcal{A}_{E_{\rho_0}}$ are of the form

$$A_{E_{\rho_0}} = \sum_i \rho_0(A_{0,i}) A_{-,i} \otimes P_{\rho_0} \otimes A_{+,i}, \quad A_{\pm,i} \in \mathcal{A}_\pm,\ A_0 \in \mathcal{A}_0.$$

The scalar factor $\mathrm{e}^{-\beta H_0^{\rho_0}}$ cancels out in the definition of the Gibbs states

$$\Longrightarrow \omega_{\rho_0,\beta}(A_{E_{\rho_0}}) = \sum_i \rho_0(A_{0,i})\omega^-_{\rho_0,\beta}(A_{-,i})\omega^+_{\rho_0,\beta}(A_{+,i}),$$

that is, $\omega_{\rho_0,\beta} = \omega^-_{\rho_0,\beta} \otimes \rho_0 \otimes \omega^+_{\rho_0,\beta}$ on $\mathcal{A}_{E_{\rho_0}}$ is the Zeno-Gibbs equilibrium.

Example: The X-Y Model 3/3

Physical implications:

$\Longrightarrow$ $\omega_{\rho_0,\beta}$ is the unique $(\tau^{E_{\rho_0}}, \beta)$-KMS state on $\mathcal{A}_{E_{\rho_0}}$.

$\Longrightarrow$ $\tau^{E_{\rho_0}}$ decouples the left and right subchains.

$\Longrightarrow$ $H_{\pm}^{\rho_0}$ are averaged Hamiltonians w.r.t. $\rho_0 \Longrightarrow$ the Zeno dynamics imposes **boundary conditions** on the lateral subchains, parametrized by the single complex number $\rho_0(a_0)$

$\Longrightarrow$ The difference $H - E_{\rho_0} H E_{\rho_0}$ is a finite combination of a_x, a_x^*, $x = 0, \ \pm 1$.

$\Longrightarrow$ $H - E_{\rho_0} H E_{\rho_0}$ is bounded, and moreover entire analytic for $\tau^{E_{\rho_0}}$.

$\Longrightarrow$ (Return to equilibrium) The system prepared in the (τ, β)-KMS state will evolve to the Zeno equilibrium $\omega_{\rho_0,\beta}$, under $\tau^{E_{\rho_0}}$.

$\Longrightarrow$ Observation of the state of a single lattice site changes the global equilibrium.

Summary and Conclusions

- The abstract treatment of the Zeno paradox (Misra & Sudarshan) was transferred to the context of quantum statistical mechanics (von Neumann algebras, KMS states).
- Zeno dynamics operates naturally on $\mathcal{A}_E = E\mathcal{A}E$.
- A sufficient short-time asymptotic condition (AZC) for the emergence of Zeno dynamics was given.
- (AZC) was shown to be effective:
  - Fundamentally, in the prevention of return to equilibrium, and
  - concretely, in the generic class of models of quantum spin systems.
- The generator of Zeno dynamics can be identified as EHE,
- which allows one to construct Gibbs equilibria for it.
- Zeno dynamics can impose boundary conditions on a quantum spin system.
- The treatment of the X-Y model exhibited new phenomenological aspects:
  - A microscopic measurement of a quantum state (at a single site) changes the global equilibrium.
  - The system spontaneously evolves to the Zeno equilibrium.

Ref's: A U Schmidt (2002) J. Phys. A **35** 7817–7825, `math-ph/0203008`,
and (2003) J. Phys. A **36** 1135–1148, `math-ph/0207013`